Transition to synchronization in a Kuramoto model with the first- and second-order interaction terms

Keren Li, Shen Ma, Haihong Li,* and Junzhong Yang†

School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, People's Republic of China

(Received 8 September 2013; revised manuscript received 30 December 2013; published 24 March 2014)

We investigate a Kuramoto model incorporated with the first-order and the second-order interaction terms. We show that the model displays the coexistence of multiattractors and different attractors may be characterized by the phase distributions of oscillators. By investigating the transition diagrams in both forward continuation and backward continuation, we find that the synchronous state with unimodal phase distribution is the most stable one while the state in cluster synchrony with evenly distributed bimodal phase distribution is the least stable one. We also present the phase diagram of the model in the parameter space.

DOI: 10.1103/PhysRevE.89.032917 PACS number(s): 05.45.Xt, 89.75.−k

I. INTRODUCTION

Synchronization, a phenomenon since the early days in physics, is found in systems such as fireflies flashing in unison [1,2], Josephson junction arrays [3], atomic recoil lasers [4], electrochemical oscillators [5], applauding persons in a large audience [6], pedestrians on footbridges [7], and many others. The Kuramoto model (KM) has become a paradigmatic model in the study of synchronization since it is proposed by Kuramoto in 1975 [8],

The original KM consists of N phase oscillators. Each oscillator has its own natural frequency ω drawn from a given probability density $g(\omega)$ and interacts with the mean field with a global coupling strength K. Synchronization is observed when the coupling strength K is above a threshold K_c and the synchronization transition is a continuous one [9]. Several generalizations to the original KM have been made. In an original KM, the coupling strength K is assumed to be positive to account for an attractive interaction. Tsimring *et al.* considered the case in which the interaction between oscillators and the mean field is a repulsive one ($K < 0$) and they found that synchronization fails for an array of nonidentical phase oscillators provided that the number of oscillators is sufficiently large [10]. Hong and Strogatz studied the situation in which the coupling strength is treated as an oscillator's ability reacting to the mean field individually [11,12]. In their works, both positive and negative coupling strengths are present in the population. They found a novel travelling wave state when incoherence state becomes unstable. With the well recognition of complex networks in natural and social systems in the past decade, the requirement of global interaction among oscillators has been discarded and phase oscillators are assumed to sit on nodes on networks and to interact only with their neighbors. Some interesting findings have been made such as the explosive synchronization when the natural frequencies of oscillators are positively correlated to the numbers of neighbors of oscillators [13,14] and dependence of transition scenarios on the topological properties of networks.

Kuramoto showed [9] that the interaction between phase oscillators should take the general form of $H(\theta_i - \theta_j)$, where θ_i and θ_j are the phases of oscillators i and j and H is a 2π-periodic function. However, most of works on KM only involve with the interaction taking the form $\sin(\theta_i - \theta_j)$, which is the first order of H in a Fourier expansion. Recently, Engelbrecht and Mirollo showed [15] that the long-term average frequency as a natural frequency displays a devil staircase when an additional second-order interaction term is introduced to the original KM. Tanaka and Aoyagi studied the KM model with the addition of three-body interaction which actually leads to a second-order interaction term [16]. They found the coexistsence of multiattractors. Using a variation of recent dimensionality-reduction technique of Ott and Antonsen [17], Skardal *et al.* studied coupled phase oscillators with a single higher-order coupling [18] and characterized the cluster synchrony in the system. In a nonlocally KM, Battogtokh found that phase turbulence persists across a wide interval of coupling ranges if both the first and the second coupling terms are taken into consideration while it exists only at long coupling range if only the first-order coupling is present [19]. In this work, we will investigate the synchronization transition in the KM with both the first- and the second-order interaction terms.

II. MODEL

The model is described as

$$\dot{\theta}_i = \omega_i + \frac{K_1}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) + \frac{K_2}{N}\sum_{j=1}^{N}\sin[2(\theta_j - \theta_i)], \quad i = 1,2\ldots N, \qquad (1)$$

where $\theta_i(t)$ is the phase of the ith oscillator at time t and N is the number of phase oscillators in the system. ω_i is the natural frequency of the ith oscillator and is chosen at random from a Lorentzian probability density $g(\omega) = \gamma/[\pi(\gamma^2 + \omega^2)]$ of width $\gamma = 0.5$. K_1 and K_2 are the coupling strengthes for the interactions through the first-order and the second-order interaction terms.

*haihongli@bupt.edu.cn
†jzyang@bupt.edu.cn

We define two order parameters

$$Z_k = R_k e^{ik\Phi_k} = \frac{1}{N}\sum_{j=1}^{N} e^{ik\theta_j}, \quad k = 1,2. \tag{2}$$

Then, by introducing $\phi_i = \theta_i - \Phi_1$, Eq. (1) can be rewritten in terms of these order parameters

$$\dot{\phi}_i = \omega_i - \Omega_1 - K_1 R_1 \sin(\phi_i) - K_2 R_2 \sin[2(\phi_i - \Delta\Phi)], \quad i = 1,2\ldots N, \tag{3}$$

where $\Delta\Phi = \Phi_2 - \Phi_1$ and $\Omega_1 = d\Phi_1/dt$ is the mean-field frequency. In the case with $K_1 = 0$ (or $K_2 = 0$), the model is reduced to the original KM and the incoherent state becomes unstable above $K_i = 1 (i = 1,2)$.

In model (1), the extent of the synchronization is better reflected by the order parameter Z_2. However, as shown in the following, though the order parameter Z_1 is not responsible for the onset of synchronization, it is still an important measure reflecting the organization of oscillators in a microscopic view.

For a general case with nonzero K_1 and nonzero K_2, the dynamics in model (1) strongly depends on the initial conditions. To get a clear picture on it, we assume that partial synchronization has been built up and oscillators with natural frequency ω could be synchronized. Then the phase ϕ^* of the synchronized oscillators satisfies

$$\frac{\omega}{K_1 R_1} = \sin(\phi^*) + \frac{K_2 R_2}{K_1 R_1}\sin[2(\phi^* - \Delta\Phi)]. \tag{4}$$

The function on the right-hand side of Eq. (4) is dependent of K_2R_2/K_1R_1 and $\Delta\Phi$. Figure 1(a) shows the results at $\Delta\Phi = 0$. For $K_2R_2/K_1R_1 > 0.5$, there may exist four ϕ^* to satisfy Eq. (4) provided that ω is sufficiently close to $\omega = 0$: two stable equilibria and two unstable equilibria. The distance between two stable (or unstable) equilibria is around π. On the other hand, only two ϕ^* can be found for $K_2R_2/K_1R_1 < 0.5$ and only one of them is stable. When $K_2R_2/K_1R_1 > 0.5$, the existence of two stable ϕ^* for the oscillator with a proper natural frequency indicates that which ϕ^* is taken by the oscillator in the evolution depends on its initial condition and on the history of the system. Consequently, for arbitrary initial conditions, the model may develop into states in cluster synchrony where oscillators are divided into two groups which are separated from each other with a distance around π along the space of ϕ. Figures 1(b)–1(d) show that the function at several $\Delta\Phi$ for different K_2R_2/K_1R_1. The number of solutions to Eq. (4) is unchanged for both low and high K_2R_2/K_1R_1. However, the situation will become complicated at $K_2R_2/K_1R_1 \simeq 0.5$, where the number of ϕ^* strongly depends on $\Delta\Phi$. Fortunately, in the numerical investigations of model (1), we always find $\Delta\Phi \simeq 0$ and, consequently, K_2R_2/K_1R_1 will be the only quantity that matters.

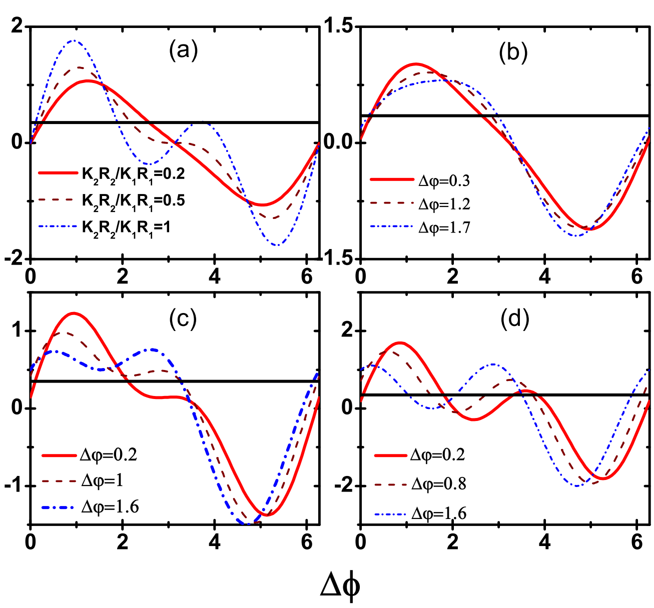

FIG. 1. (Color online) The function on the right-hand side of Eq. (4) is plotted against ϕ for different K_2R_2/K_1R_1 and different $\Delta\Phi$. (a) $\Delta\Phi = 0$; (b) $K_2R_2/K_1R_1 = 0.2$; (c) $K_2R_2/K_1R_1 = 0.5$; (d) $K_2R_2/K_1R_1 = 1$. The black horizontal line indicates a possible ω/K_1R_1 which may intersect with the function once or twice.

III. NUMERICAL RESULTS

We numerically investigate the dynamics in model (1) by a fourth-order Runge-Kutta algorithm with a time step $\delta t = 0.01$ and, throughout the work, we let $N = 10\,000$.

First, we investigate the effects of the initial conditions on the dynamics of the system. To do it, we initially assign each oscillator a phase randomly drawn from the distribution $p\delta(\theta) + (1-p)\delta(\theta - \pi)$. p denotes the fraction of oscillators adopting $\theta = 0$ as their initial phases. Due to the symmetry, we only consider p in the range of $p \in [0,0.5]$. In the case with $K_1 = 0$, the initial conditions with $p = 0$ always develops into a phase distribution with a single peak and those with $p = 0.5$ will always evolve to a state in cluster synchrony with two peaks with the same height. However, for nonzero K_1 and K_2, the final phase distribution is dependent on not only p but also on K_1 and K_2. Note that we also simulate the model by initializing the phase oscillators with two smooth peak distributions with a nonzero width, such as two Gaussians or two Lorentzians, and we find that simulation results are similar to those using two δ functions.

Figure 2(a) shows the results for $K_1 = 0.8$ and $K_2 = 0.6$. The amplitude of the order parameters R_1 and R_2 are plotted against p. From the data, we know that $K_2R_2/K_1R_1 \simeq 0.5$ no matter what p is, which is a marginal case between single stable ϕ^* and two stable ϕ^*. As a result, the phase distribution always looks like a single-peak one, which suggests that R_1 is always higher than R_2 and that both R_1 and R_2 are insensitive to p. Then we increase K_2. Figure 2(b) shows the case for $K_1 = 0.8$ and $K_2 = 1$. In this case, both R_1 and R_2 show strong dependence on p. The feature can be understood by noticing that K_2R_2/K_1R_1 is always higher than 1. For K_2R_2/K_1R_1 much high than 0.5, the phase distribution with two peaks becomes possible with a proper preparation of initial conditions. As shown by the insets, the phase distribution is changed from a structure with a single peak to a one with two peaks as p increases and the the structure with two peaks becomes more prominent when p increases toward 0.5. For larger K_2R_2/K_1R_1, for example, in Fig. 2(c) where $K_2 = 1.2$, we may find that the phase distribution could reach one having two peaks with the same height. For a state in cluster synchrony

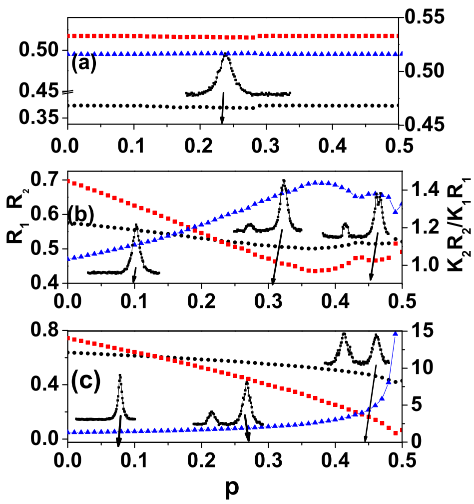

FIG. 2. (Color online) The amplitudes of order parameters R_1 (red square), R_2 (black circle), and the quantity R_2K_2/R_1K_1 (blue triangle) are plotted against p for different K_2 at $K_1 = 0.8$. (a) $K_2 = 0.6$, (b) $K_2 = 1$, and (c) $K_2 = 1.2$. Insets show the phase distributions of oscillators in steady states at different K_2 denoted by arrows.

where the phase distribution with two similar peaks has been built up, R_1 approaches zero since the two stable ϕ^* are separated from each other at around π.

The results in Fig. 2 suggest the coexistence of an infinite number of states in cluster synchrony which are characterized by their different phase distributions. Now two questions arise: How stable are these states with different phase distributions? How do these states depend on the parameters K_1 and K_2? To answer the first question, we investigate how a given state in cluster synchrony responses to perturbation. We first prepare a state in cluster synchrony at certain K_1, K_2, and p. Then we perturb the state at a certain time. The perturbation applying to the phase of every oscillator takes the form of $\theta \to \theta + \xi$, where ξ is a random number uniformly distributing in the range of $(0,\alpha)$. The response of the given state to perturbation is measured by ΔR_1 and ΔR_2, the difference between the amplitudes of order parameters before and after the perturbation. Figure 3 presents ΔR_1 and ΔR_2 against α for different combinations of K_1, K_2, and p where each datum is acquired by averaging over 10 different realizations of ξ. Interestingly, there exists a threshold on the strength of perturbation for each set of parameters. Below the threshold, ΔR_1 and ΔR_2 stay at constants, which suggests that the state is linearly stable (nonzero ΔR_1 and ΔR_2 could be caused by a finite number of oscillators, such as larger N and lower ΔR_1 and ΔR_2). In contrast, ΔR_1 and ΔR_2 strongly depend on the strength of perturbation when α is higher than the threshold, which indicates that perturbation brings the model into other different states. Though the critical strength of perturbation is always low and distinguishing different states in cluster synchrony only by the order parameters is not sufficient, the existence of a threshold on the strength of perturbation in Fig. 3 suggests that the observed states in cluster synchrony are asymptotically stable but with much narrower attraction basins.

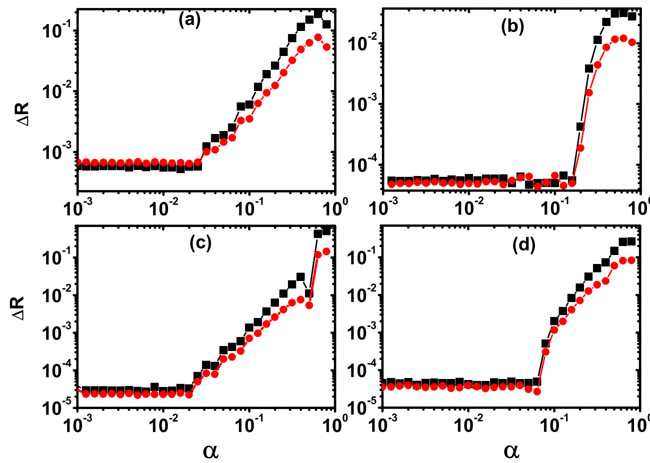

FIG. 3. (Color online) The responses ΔR_1 (black square) and ΔR_2 (red circle) of states in cluster synchrony to different strength α of perturbation. (a) $K_1 = 0.75$, $K_1 = 0.75$, and $p = 0.1$; (b) $K_1 = 0.75$, $K_1 = 0.75$, and $p = 0.3$; (c) $K_1 = 0.6$, $K_1 = 1.2$, and $p = 0.1$; (d) $K_1 = 0.6$, $K_1 = 1.2$, and $p = 0.3$.

To investigate how the states in cluster synchrony depend on the parameters K_1 and K_2, we consider two types of transition diagrams, labeled as forward continuation and backward continuation. The coupling strength are successively increased (or decreased) by a $\delta\lambda$ in the forward (or backward) continuation and the initial conditions for one coupling strength are the final state of the previous one. Both the forward continuation and the backward continuation start with the initial conditions with $p = 0.5$. In Fig. 4, we show the amplitudes of the order parameters R_1 and R_2 against K_2 for different K_1.

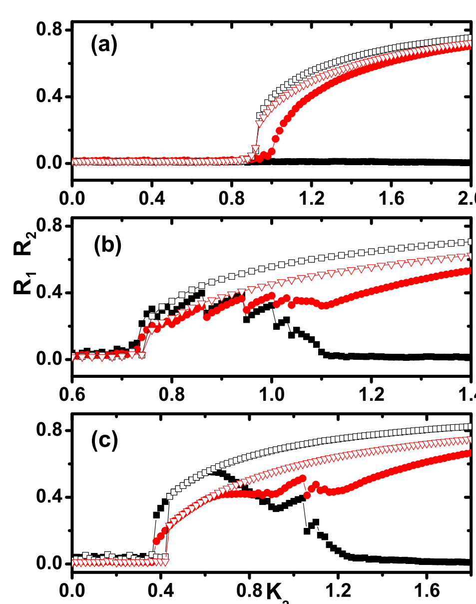

FIG. 4. (Color online) The amplitudes of order parameters R_1 (black square) and R_2 (red circle) are plotted against K_2 at different K_1. (a) $K_1 = 0.2$, (b) $K_1 = 0.4$, and (c) $K_1 = 0.85$. The transition diagram in the forward continuation (or the backward continuation) is denoted by open symbols (or solid symbols).

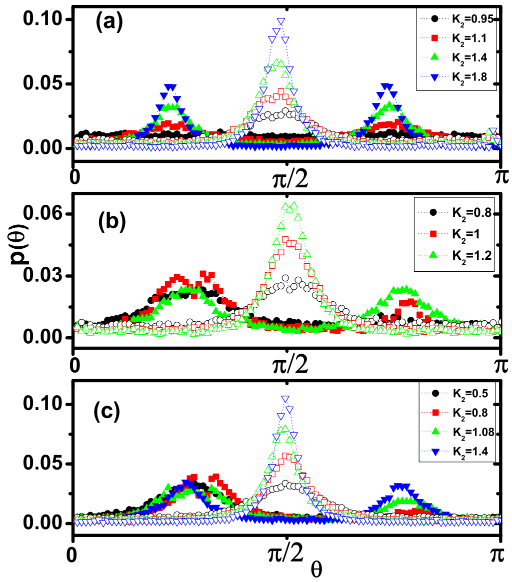

FIG. 5. (Color online) The phase distributions in steady states for different K_2 at (a) $K_1 = 0.2$, (b) $K_1 = 0.4$, and (c) $K_1 = 0.85$ in the forward continuation (in open symbols) and the backward continuation (in solid symbols).

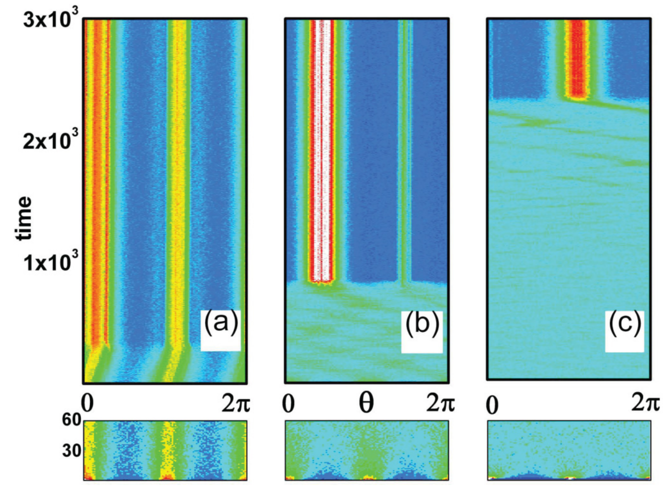

FIG. 6. (Color online) The time evolutions of phase distribution for different K_2 at $K_1 = 0.4$. The initial phase distribution is an evenly distributed bimodal phase distribution. The top panels show the evolution in a long time run and the bottom panels show the transient in a short time interval. (a) $K_2 = 1.07$; (b) $K_2 = 0.97$; (c) $K_2 = 0.8$.

The transition scenarios in these two types of continuations differ substantially. For the forward continuation, the transition scenario seems to be continuous except for the situation with high K_1 in which the transition between the incoherent state and a partially synchronized state is a sharp one. In contrast, the transition scenario in the backward continuation is always discontinuous. In the parameter range we investigated, there may exist several regimes. From one regime to another, R_1 and R_2 show large jumps. The number of regimes strongly depends on the coupling strength K_1. For example, there are no jumps on the curves of R_1 and R_2 as functions of K_2 at $K_1 = 0.2$. As shown in Fig. 4(a), R_1 stays at zero and R_2 reduces to zero gradually until $K_2 = 1$. However, in Fig. 4(b), where $K_1 = 0.4$, and Fig. 4(c), where $K_1 = 0.85$, we find the jumps on the curves of both R_2 and R_1 with the change of K_2. To be noted, as shown in Fig. 4, the deviation of R_1 from $R_1 = 0$ means more than the transition between the incoherent state and a partially synchronized state—it could signify the instability of states in cluster synchrony with evenly distributed phase distributions.

The transition scenarios shown in Fig. 4 can be understood by the variation in the phase distributions of oscillators. Figure 5(a) shows the phase distributions at $K_1 = 0.2$ for different K_2 in both the forward continuation and the backward continuation. Clearly, in the presence of synchronization, the qualitative properties of the phase distributions are unchanged in both the forward continuation and the backward continuation when K_2 varies: the phase distributions are always unimodal in the forward continuation while those in the backward continuation are always evenly distributed bimodal one. On the contrary, things differ substantially in the backward continuation at $K_1 = 0.4$ and $K_1 = 0.85$ though the phase distributions in the forward continuation are still the unimodal ones. As shown in Figs. 5(b) and 5(c), the phase distributions manifest themselves as the evenly distributed bimodal ones for large K_2. However, with the decrease of K_2, we find that the evenly distributed bimodal phase distributions are replaced by unevenly distributed bimodal ones. More lower K_2, more uneven the phase distribution. Moreover, when the transitions between the incoherent state and the partially synchronized states are approached, the bimodal phase distributions transit to unimodal ones. For example, the phase distributions at $K_2 = 0.8$ in Fig. 4(b) and at $k_2 = 0.5$ in Fig. 4(c) are the unimodal ones.

The results in Figs. 4 and 5 can be summarized as following. Partially synchronized states are characterized by their phase distributions and the stabilities of states in cluster synchrony are strongly dependent on their phase distributions. The states with evenly distributed bimodal phase distributions are the least stable ones which become unstable first in the backward continuation. The states with unimodal phase distributions are the most stable ones, which can also be reflected by the fact that the states are the ones developed directly from the incoherent state in the forward continuation. The states with unevenly distributed bimodal phase distributions are the intermediate ones. Generally, the more uneven their phase distributions the more stable they are.

To get more intuitions on the transitions in the backward continuation, we investigate how the phase distributions of oscillators evolve with time. We first prepare an evenly distributed bimodal phase distribution which is evolved from the initial condition with $p = 0.5$ at $K_1 = 0.4$ and $K_2 = 1.2$. Then the prepared state is used as the initial condition for different K_2 at $K_1 = 0.4$. In Fig. 6(a) where $K_2 = 1.07$, the final phase distribution is an unevenly distributed bimodal one and the final distribution is developed from the initial evenly distributed bimodal one continuously. On the other hand, Fig. 6(b) shows that, at $K_2 = 0.97$, the final unevenly distributed bimodal phase distribution cannot be developed from the initial evenly distributed one directly. The initial

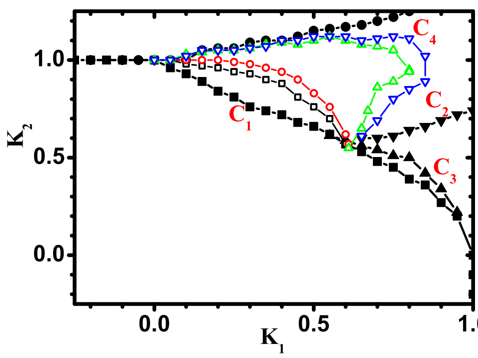

FIG. 7. (Color online) The phase diagram of model (1) on the space of K_1 and K_2. The curve C_1 (black solid square) denotes the boundary between the incoherent state and partially synchronized states. Below the curve, no synchronous state exists. The curve C_2 (black solid circle) denotes the boundary between the multiattractors and single attractor. The regimes enclosed by the curves C_1 and C_3 denotes the coexistence between the incoherent state and the synchronous states. Below the curve C_4 the synchronous states with evenly distributed phase distribution are unstable. The curves with open symbols denote the discontinuous transition in the backward continuations for different initial phase distributions with $p = 0.1$ (black open square), $p = 0.2$ (red open circle), $p = 0.4$ (green open up-triangle), and $p = 0.45$ (blue open down-triangle).

bimodal distribution first evolves into a seemly incoherent transient state [see the bottom panel in Fig. 6(b)] and then the final distribution grows up out of the incoherent transient. For lower K_2 such as $K_2 = 0.8$, Fig. 6(c) shows that the building up of a unimodal phase distribution also undergoes a transient period of irregular dynamics. Whether the phase distribution for a state in cluster synchrony can be developed directly from the one of another state determines whether the transition between them is a continuous one, which accounts for the discontinuities exited in the transition scenario for some combinations of K_1 and K_2 in Fig. 4. To be mentioned, whether a state in cluster synchrony can be developed directly from another one is possibly determined by the difference between their phase distributions, which serves as a perturbation. When the perturbation is strong, the transient incoherence seems likely. However, more thorough investigations are required to reach a convincing conclusion, which is beyond the scope of this work.

To get an overview on the dynamics in model (1), we present the phase diagram of model (1) on the plane of K_1 and K_2 in Fig. 7. There are four curves dividing the plane of K_1 and K_2 into several regimes. Below the curve C_1, only incoherent states can be found. Below the curve C_2, the coexistence of infinite number of states in cluster synchrony is lost and only one synchronous state with unimodal phase distribution exists. The incoherent state coexists with synchronous state in the regime enclosed by the curves C_1 and C_3. The curve C_4 is acquired by the transition between the states in cluster synchrony with $R_1 = 0$ and with $R_1 \neq 0$, below which the states with evenly distributed phase distribution become unstable. We also present in Fig. 7 other four curves which are labeled by different p. These curves locate the first discontinuous transition in the backward continuations for both K_2 and K_1 when the initial conditions are characterized by p. Clearly, on the right side of all of these curves, a state developed from arbitrary initial condition can continuously approach the incoherent state by decreasing K_1 and K_2 along proper chosen pathes.

Tsimring *et al.* considered an variant of the original KM in which the interaction between oscillators and the mean field is a repulsive one [10]. They found that synchronization fails for a population of nonidentical phase oscillators provided that the number of oscillators is sufficiently large. Here, we present the results for the model with negative K_1 or negative K_2 in Fig. 7. Interestingly, we find that the incoherent state always becomes unstable at $K_2 \geqslant 1$ (or $K_1 \geqslant 1$) for negative K_1 (or negative K_2). The finding indicates that only the coupling term with positive coupling strength is relevant to the instability of the incoherent state. A possible explanation is that the coupling term with negative coupling strength just act as a noise which can impacts on the coherent state in the model only when they are strong enough.

IV. DISCUSSIONS

In a recently published elegant work [20], Komarov and Pikovsky also studied model (1). They provided the solutions for stationary synchronous states based on self-consistency equations and the diagram of different states in the plane of K_1 and K_2. In their work, the synchronous states are characterized by a parameter σ describing the distribution of oscillators' phases over different branches (or different stable ϕ^* as discussed in Sec. II) and σ is assumed to be independent of the natural frequencies ω of oscillators. Though the synchronous states in model (1) cannot be determined only by a parameter σ and σ should be dependent of ω, their results still shed great insight on the dynamics in model (1).

Though the model we studied in this work is the same as the one used by Komarov and Pikovsky, we investigated it in a different way and drew some conclusions they did not discuss and even ones that differed from theirs. First, Komarov and Pikovsky claimed in their work that the states in cluster synchrony in the parameter region beyond L_2 (see Fig. 2 in their work) are linearly neutrally stable. However, in our work, those states in cluster synchrony are asymptotically stable though with a much narrower attraction basin (see Fig. 3). Second, Komarov and Pikovsky presented the parameter regimes for different states in cluster synchrony using a self-consistence approach. However, they did not pay much attention to the transition scenario. In our work, we discussed the transition scenario and found that the scenario strongly depends on the parameter. Notably, we found that the transition from one synchronous state to another with the change of the parameter could be discontinuous or continuous. We also presented a possible explanation for it. Third, Komarov and Pikovsky characterized the different states in cluster synchrony by a parameter σ assumed to be independent of the natural frequency of oscillators. However, whether such an assumption holds for arbitrary initial conditions is questionable. In our work, we characterized the states by the asymmetry in their final phase distributions. We found that the states in cluster synchrony with evenly distributed bimodal phase distribution are the least stable ones, which is not found by Komarov and

Pikovsky ($\sigma = 0.5$ amounts to the state with evenly distributed bimodal phase distribution in their work). Last, we explored a larger parameter regime in the plane of K_1 and k_2. We found that the onset of synchronization is the same as the model with only the first-order or the second-order interaction term if one of K_1 and K_2 is negative. In summary, we think that the results in our work could be complementary to the work by Komarov and Pikovsky.

ACKNOWLEDGMENT

This work was supported by Grant No. 71301012 from the Chinese Natural Science Foundation.

[1] J. Buck and E. Buck, Sci. Am. **234**, 74 (1976).

[2] J. Buck, Quart. Rev. Biol. **63**, 265 (1988).

[3] J. W. Swift, S. H. Strogatz, and K. Wiesenfeld, Physica D **55**, 239 (1992); K. Wiesenfeld, P. Colet, and S. H. Strogatz, Phys. Rev. Lett. **76**, 404 (1996); Phys. Rev. E **57**, 1563 (1998).

[4] J. Javaloyes, M. Perrin, and A. Politi, Phys. Rev. E **78**, 011108 (2008).

[5] I. Z. Kiss, W. Wang, and J. L. Hudson, Chaos **12**, 252 (2002); M. Wickramasinghe and I. Z. Kiss, Phys. Rev. E **83**, 016210 (2011).

[6] Z. Néda, E. Ravasz, T. Vicsek, Y. Brechet, and A. L. Barabási, Phys. Rev. E **61**, 6987 (2000).

[7] B. Eckhardt, E. Ott, S. H. Strogatz, D. M. Abrams, and A. McRobie, Phys. Rev. E **75**, 021110 (2007).

[8] Y. Kuramoto, in *International Symposium on Mathematical Problems in Theoretical Physics*, Lecture Notes in Physics Vol. 39, edited by H. Araki (Springer, New York, 1975).

[9] Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence* (Springer, New York, 1984).

[10] L. S. Tsimring, N. F. Rulkov, M. L. Larsen, and M. Gabbay, Phys. Rev. Lett. **95**, 014101 (2005).

[11] H. Hong and S. H. Strogatz, Phys. Rev. Lett. **106**, 054102 (2011).

[12] H. Hong and S. H. Strogatz, Phys. Rev. E **84**, 046202 (2011).

[13] J. Gómez-Gardeñes, S. Gómez, A. Arenas, and Y. Moreno, Phys. Rev. Lett. **106**, 128701 (2011).

[14] I. Leyva, R. Sevilla-Escoboza, J. M. Buldú, I. Sendiña-Nadal, J. Gómez-Gardeñes, A. Arenas, Y. Moreno, S. Gómez, R. Jaimes-Reátegui, and S. Boccaletti, Phys. Rev. Lett. **108**, 168702 (2012).

[15] J. R. Engelbrecht and R. Mirollo, Phys. Rev. Lett. **109**, 034103 (2012).

[16] T. Tanaka and T. Aoyagi, Phys. Rev. Lett. **106**, 224101 (2011).

[17] E. Ott and T. M. Antonsen, Chaos **18**, 037113 (2008).

[18] P. S. Skardal, E. Ott, and J. G. Restrepo, Phys. Rev. E **84**, 036208 (2011).

[19] D. Battogtokh, Phys. Lett. A **299**, 558 (2002).

[20] M. Komarov and A. Pikovsky, Phys. Rev. Lett. **111**, 204101 (2013).